MCMILLAN–MCGEE: MODELLING INDUCTION HEATER

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Abstract. In this project we consider a cylindrical induction heater of the type built by the McMillan–McGee corporation. Our goal is to develop a model for the electric field induced inside the heater, and understand the effects of changing materials and physical dimensions. We studied the electrical field intensity through the Maxwell equations, which allowed us to find an exact expression for the field intensity and calculate the power flowing into the casing of the pipe.

1. Problem Description

Contaminated soils are a significant environmental and safety concern. Many contaminants have the ability to flow into aquifer systems, thereby contaminating the public water supply. The depth at which some contaminants occur renders the use of excavation prohibitively expensive. Therefore, other methods are employed to remove contaminants in-situ, where depth is not a factor.

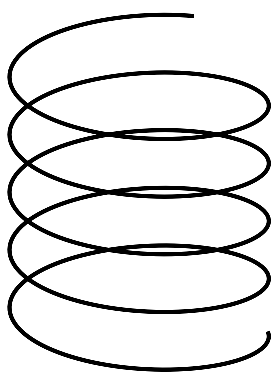
One such method includes heating the soil using electricity in order to vaporize the contaminants, which are subsequently extracted from the soil. Methods of heating the soil are also utilized in connection with heating subterranean heavy oil reservoirs or bitumen deposits to reduce the viscosity of the hydrocarbons so that it can be recovered more easily. A typical approach is to bury a tall cylindrical heating element in the ground, and thus heat it up at the desired depth.

However, cost considerations limit the heating elements and their housings to a small diameter. Moreover, the heating equipment used in such operations are sunk costs, as they are typically left in the ground after a remediation project is completed. Therefore, there is a need for an economical method and a device for heating soil that provides a large heating surface area, enables the selective heating of vertical extents of the element to different temperatures, and is capable of achieving soil temperatures sufficient to remediate contaminants with high boiling points, while allowing for recovery of at least some of the heating equipment after operations have concluded.

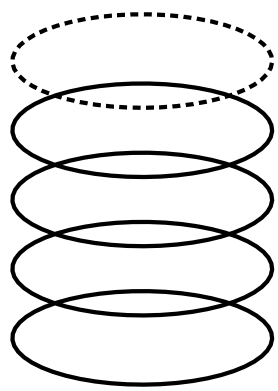
One example of such a device consists of a conductive casing, such as a steel pipe, which is heated by the induction of an electric current within its wall as a result of the passing of an alternating current through a conductor located inside the casing. The alternating current has to be of a sufficiently high frequency in order to exploit the skin effect: it limits the penetration of the current into the pipe wall to a very thin shell where, given a conductor in helical form, the current flows in a circumferential direction. That is, the current density in the casing is greatest near the inner surface of the casing. The skin effect results in establishing an appreciable resistance in the casing, under which the passage of current through the resistance generates heat.

2. Geometry of The Problem

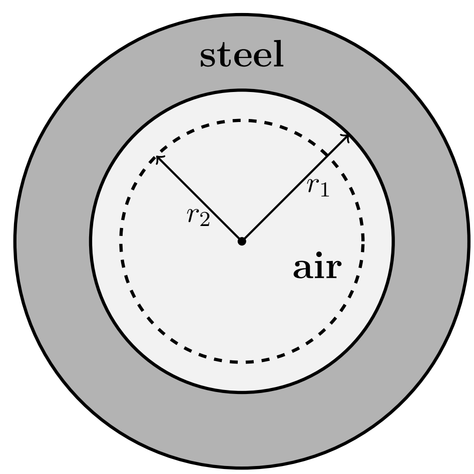
In order to simplify our model, we may assume that we have conducting rings inside the induction heater instead of a conducting helix.



Idealization 1:
conducting helix



Idealization 2:
conducting rings



View from above

3. Methodology

We start with the description of the mathematical model. The electromagnetic field is governed by the Maxwell equations, that relate the electric and magnetic field intensities to the current induced in the work coil. In this work, we assume our domain to have two distinct regions: the inside of the cylindrical shell and inside the steel housing itself. The governing equations are mathematically the same in both regions; the need for separation comes from the difference in physical parameters.

We write the equations in the cylindrical coordinates system to take advantage of the rotational symmetry. Next, we derive an elliptic partial differential equation satisfied by the angular component of the electric field intensity. We use the Fourier series representation in the vertical variable, which reduces the original PDE to a diagonal system of Bessel ODEs. Consequently, the Fourier coefficient of the solution are linear combinations of the Bessel functions. The required constants in the solution are found from the boundary conditions at the origin and at infinity, and the interface conditions between the interior and the housing of the heating element.

The second part of this work consists of numerically evaluating and visualizing the solutions. Moreover, we are able to evaluate the power flowing into the casing using the Poynting vector field.

4. Setup

The electromagnetic fields are described by the Maxwell's equations:

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H}, \tag{4.1a}$$

$$\nabla \times \mathbf{H} = i\omega\epsilon\mathbf{E} + \mathbf{J}, \tag{4.1b}$$

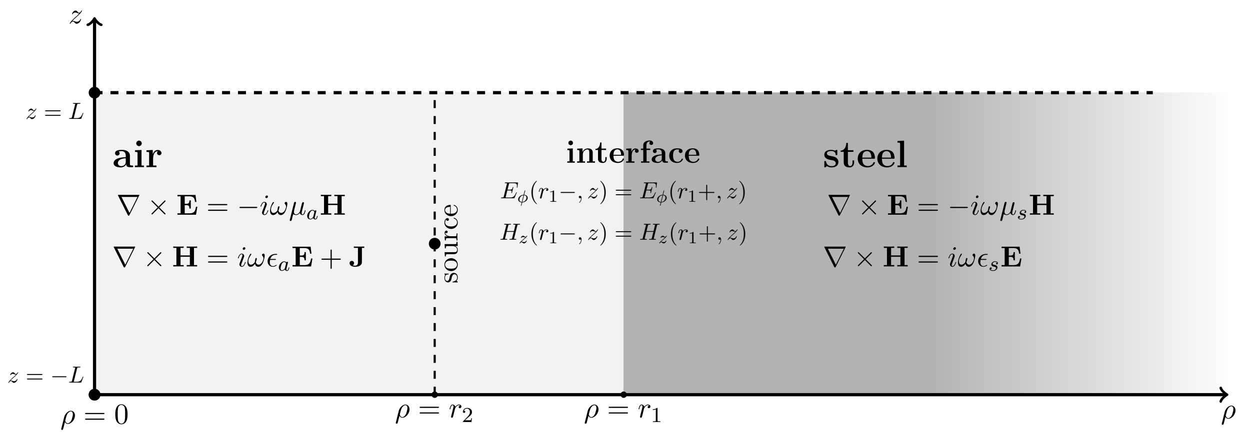
where

- $\mathbf{E}$ is the electric field intensity (volts/meter)
- $\mathbf{H}$ is the magnetic field intensity (amps/meter)
- $\mathbf{J}$ is the electric current density (amps/meter^2)
- μ is the permeability of the medium (henrys/meter)
- ϵ is the permittivity of the medium (farads/meter).

We take the current source to be an infinite collection of rings of radius r_2, spaced at the intervals of $2L$. We will work with a single loop located at $z = 0$, and impose periodic boundary conditions in $z \in [-L, L]$. The current density is represented as

$$\mathbf{J}(\rho, z) = \hat{\boldsymbol{\phi}}\, I_0 \frac{\delta(\rho - r_2)\delta(z)}{2\pi\rho}, \tag{4.2}$$

where I_0 is the total current (constant, in amps). We will solve a coupled system of PDEs: both are (4.1), but with different values of the physical parameters μ and ϵ. The setup is summarized in the figure below.



Boundary conditions: E_ϕ is finite at $\rho = 0$, vanishes as $\rho \to \infty$ and $2L$-periodic in z.

5. Symmetry Reduction

Since the system is invariant under rotations, we have $\partial_\phi = 0$ for both regions. We write out the Maxwell's system (4.1) in components, and reduce it:

$$\frac{1}{\rho}\cancel{\frac{\partial E_z}{\partial \phi}} - \frac{\partial E_\phi}{\partial z} = -i\omega\mu_a H_\rho \tag{5.1a}$$

$$\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = -i\omega\mu_a H_\phi \tag{5.1b}$$

$$\frac{1}{\rho}\left(\frac{\partial}{\partial \rho}\big(\rho E_\phi\big) - \cancel{\frac{\partial E_\rho}{\partial \phi}}\right) = -i\omega\mu_a H_z \tag{5.1c}$$

$$\frac{1}{\rho}\cancel{\frac{\partial H_z}{\partial \phi}} - \frac{\partial H_\phi}{\partial z} = i\omega\epsilon_a E_\rho \tag{5.1d}$$

$$\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = i\omega\epsilon_a E_\phi + I_0\frac{\delta(\rho - r_2)\delta(z)}{2\pi\rho} \tag{5.1e}$$

$$\frac{1}{\rho}\left(\frac{\partial}{\partial \rho}\big(\rho H_\phi\big) - \cancel{\frac{\partial H_\rho}{\partial \phi}}\right) = i\omega\epsilon_a E_z \tag{5.1f}$$

The system (5.1) decouples into two independent systems: $\big[$(5.1a), (5.1c), (5.1e)$\big]$ and $\big[$(5.1d), (5.1f), (5.1b)$\big]$. The latter lacks a source, so the solution must be zero:

$$E_\rho = E_z = H_\phi = 0.$$

The former is actually nontrivial, and will be the subject of the work in the following sections.

6. Solving The Equation: Inner Domain

Differentiating (5.1a) and (5.1c) with respect to z and ρ respectively, and substituting the results into (5.1e) gives

$$\frac{\partial^2 E_\phi}{\partial z^2} + \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\phi)\right) = -\omega^2 \mu_a \epsilon_a E_\phi + \frac{i\omega\mu_a I_0}{2\pi}\frac{\delta(\rho - r_2)\delta(z)}{\rho} \tag{6.1}$$

We set

$$\alpha = \frac{i\omega\mu_a I_0}{4\pi L} \qquad g = \frac{E_\phi}{\alpha} \qquad k_a = \omega\sqrt{\mu_a\epsilon_a} \tag{6.2}$$

and write

$$\frac{\partial^2 g}{\partial z^2} + \frac{\partial^2 g}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial g}{\partial \rho} + \left(k_a^2 - \frac{1}{\rho^2}\right) g = \frac{2L\delta(\rho - r_2)\delta(z)}{\rho}. \tag{6.3}$$

Our next step is to take the Fourier series in the z-variable (this will automatically enforce periodicity):

$$g(\rho, z) = \sum_{n\in\mathbb{Z}} \hat{g}_n(\rho) e^{i\pi n z/L} = \sum_{n\in\mathbb{Z}} \hat{g}_n(\rho) e^{ik_n z}, \tag{6.4}$$

where $k_n = \pi n/L$ is the Fourier wave number. Then the coefficients satisfy

$$\frac{1}{\rho^2}\left(\rho^2\frac{\partial^2}{\partial \rho^2} + \rho\frac{\partial}{\partial \rho} + \left(k_a^2 - k_n^2\right)\rho^2 - 1\right)\hat{g}_n(\rho) = \frac{\delta(\rho - r_2)}{\rho}. \tag{6.5}$$

Let $\lambda_n^2 = k_n^2 - k_a^2$, and set $x = \lambda_n \rho$ and $\hat{g}_n(\rho) = \hat{u}_n(\lambda_n\rho) = \hat{u}_n(x)$. Then the equation above becomes

$$\underbrace{\left(x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - \left(x^2 + 1\right)\right)}_{\text{modified Bessel operator of order 1}}\hat{u}_n(x) = \frac{x}{\lambda_n}\delta\left(x/\lambda_n - r_2\right). \tag{6.6}$$

For $x \neq \lambda_n r_2$, the equation above is the modified Bessel ODE of order 1, so we expect the solution to take the form

$$\hat{u}_n(x) = \begin{cases} A_n I_1(x) + B_n K_1(x), & 0 < x < \lambda_n r_2, \\ C_n I_1(x) + D_n K_1(x), & \lambda_n r_2 < x < \lambda_n r_1, \end{cases} \tag{6.7}$$

where I_1 and K_1 are the modified Bessel functions of order one, of the first and second kind respectively.

To find the constants A_n, B_n, C_n and D_n we begin with enforcing the continuity and jump conditions at $x = \lambda_n r_2$. The continuity condition dictates that

$$\left(C_n - A_n\right)I_1(\lambda_n r_2) + \left(D_n - B_n\right)K_1(\lambda_n r_2) = 0. \tag{6.8}$$

The jump condition[1] dictates that

$$\big(C_n - A_n\big)I_1'(\lambda_n r_2) + \big(D_n - B_n\big)K_1'(\lambda_n r_2) = \frac{1}{\lambda_n r_2}. \tag{6.9}$$

We can write (6.8) and (6.9) together as a linear system:

$$\underbrace{\begin{bmatrix} I_1 & K_1 \\ I_1' & K_1' \end{bmatrix}}_{=:M} \begin{pmatrix} C_n - A_n \\ D_n - B_n \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\lambda_n r_2} \end{pmatrix} \tag{6.10}$$

where it is understood that all entries of M are evaluated at $\lambda_n r_2$. We can invert M using the Wronskian identity for the Bessel functions, and thus obtain

$$\begin{pmatrix} C_n - A_n \\ D_n - B_n \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ \frac{1}{\lambda_n r_2} \end{pmatrix} = -\lambda_n r_2 \begin{bmatrix} K_1' & -K_1 \\ -I_1' & I_1 \end{bmatrix} \begin{pmatrix} 0 \\ \frac{1}{\lambda_n r_2} \end{pmatrix} = \begin{pmatrix} K_1 \\ -I_1 \end{pmatrix} \tag{6.11}$$

That is,

$$C_n - A_n = K_1(\lambda_n r_2) \qquad \text{and} \qquad D_n - B_n = -I_1(\lambda_n r_2). \tag{6.12}$$

Recall that Bessel function K_1 has a singularity at the origin, so we must set $B_n = 0$ for all. Consequently,

$$D_n = -I_1(\lambda_n r_2) \tag{6.13}$$

An additional condition on C_n and A_n will be provided by the continuity requirements at the interface $\rho = r_1$.

7. Solving The Equation: Outer Domain

We now develop the solution in the darker region $r_1 < \rho < \infty$. The symmetry reduction is identical to the one before, and so are the two resulting independent subsystems. The one featuring E_ρ, E_z and $H_\phi = 0$ is once again without a source, so we conclude that $E_\rho = E_z = H_\phi = 0$ for all domains.

Equation for the other three components is similar to (6.1), except we don't even have a delta source. In place of (6.1) we have

$$\frac{\partial^2 E_\phi}{\partial z^2} + \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\phi)\right) = -k_s^2 E_\phi, \tag{7.1}$$

where we have set $k_s = \omega\sqrt{\mu_s \epsilon_s}$. We proceed by writing the solution as a Fourier series in z:

$$E_\phi(\rho, z) = \sum_{n\in\mathbb{Z}} \hat{E}_n(\rho) e^{ik_n z}, \qquad k_n = \frac{\pi n}{L}. \tag{7.2}$$

[1]See Appendix A for the derivation

Substituting this representation into (7.1) gives

$$\left(\rho^2 \frac{\partial^2}{\partial \rho^2} + \rho \frac{\partial}{\partial \rho} + \left(k_s^2 - k_n^2\right)\rho^2 - 1\right)\hat{E}_n = 0. \tag{7.3}$$

With the change of variables $\nu_n^2 = k_n^2 - k_s^2,\ x = \nu_n \rho$ and $\hat{E}_n(\rho) = \hat{v}_n(\nu_n \rho) = \hat{v}_n(x)$, the equation above becomes

$$\left(x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - (x^2 + 1)\right)\hat{v}_n = 0, \tag{7.4}$$

which is the modified Bessel equation of order 1. Since we require the solution to vanish at infinity, we can write the solution as

$$\hat{E}_n(\rho) = \alpha F_n K_1(\nu_n \rho), \tag{7.5}$$

where F_n is the constant yet to be determined. (We don't actually have to put α into the expression above, but it makes the next step a bit nicer.)

8. Interface Conditions

We will require the electric field E_ϕ and the magnetic field H_z to be continuous at the interface $\rho = r_1$. The continuity condition $E_\phi(r_1-, z) = E_\phi(r_1+, z)$ is written as

$$\alpha \sum_{n\in\mathbb{Z}} \Big[C_n I_1(\lambda_n r_1) + D_n K_1(\lambda_n r_1)\Big] e^{ik_n z} = \sum_{n\in\mathbb{Z}} \alpha F_n K_1(\nu_n r_1) e^{ik_n z} \tag{8.1}$$

$$C_n I_1(\lambda_n r_1) + D_n K_1(\lambda_n r_1) = F_n K_1(\nu_n r_1). \tag{8.2}$$

Recall that D_n is already known from (6.13), so we have

$$C_n I_1(\lambda_n r_1) - F_n K_1(\nu_n r_1) = I_1(\lambda_n r_2) K_1(\lambda_n r_1) \tag{8.3}$$

To impose the continuity condition $H_z(r_1-, z) = H_z(r_1+, z)$, we first recall that by equation (5.1c) we have

$$H_z(r_1-, z) = \frac{i}{r_1 \omega \mu_a} \frac{\partial\big(\rho E_\phi(r_1-, z)\big)}{\partial \rho} \qquad \text{and} \qquad H_z(r_1+, z) = \frac{i}{r_1 \omega \mu_s} \frac{\partial\big(\rho E_\phi(r_1+, z)\big)}{\partial \rho},$$

so we must have

$$\frac{1}{\mu_a} \frac{\partial\big(\rho E_\phi(r_1-, z)\big)}{\partial \rho} = \frac{1}{\mu_s} \frac{\partial\big(\rho E_\phi(r_1+, z)\big)}{\partial \rho}. \tag{8.4}$$

After some algebraic manipulations and removal of the derivatives of the Bessel functions we get the following system for the coefficients C_n and F_n:

$$\underbrace{\begin{bmatrix} I_1(\lambda_n r_1) & -K_1(\nu_n r_1) \\ I_0(\lambda_n r_1) & -\gamma_n K_0(\nu_n r_1) \end{bmatrix}}_{=:W_n} \begin{pmatrix} C_n \\ F_n \end{pmatrix} = I_1(\lambda_n r_2) \begin{pmatrix} K_1(\lambda_n r_1) \\ K_0(\lambda_n r_1) \end{pmatrix} \tag{8.5}$$

where $\gamma_n = \frac{\nu_n \mu_a}{\lambda_n \mu_s}$. Unfortunately, matrix W_n defined above is not a Wronskian of the two Bessel functions. It's inverse would not simplify, and therefore it is not worthwhile to compute it by hand; instead, we will solve this system numerically for each n.

Moreover, the matrix W_n as presented above is extremely ill-conditioned. To mitigate this issue, we rewrite our solution in terms of the scaled Bessel functions

$$\tilde{I}_n(z) = e^{-|\Re z|} I_n(z) \qquad \text{and} \qquad \tilde{K}_n(z) = e^{z} K_n(z). \tag{8.6}$$

9. Transferred Power

The time-averaged Poynting vector field is given by

$$\mathbf{P} = \frac{1}{2}\Re\big(\mathbf{E} \times \mathbf{H}^*\big), \tag{9.1}$$

where * denotes the complex conjugate. One way to compute the power flowing through the boundary of the casing is to evaluate the integral

$$P_{\text{Poynt}} = 2\pi r_1 \int_{-L}^{L} (\mathbf{P}(r_1, z))_\rho \, dz. \tag{9.2}$$

Alternatively, we can assume that the current density inside the steel housing is given by $J_\phi = \sigma E_\phi$, and calculate the total power as

$$P_{E^2} = 2\pi\sigma \int_{-L}^{L} \int_{r_1}^{\infty} E_\phi^2 \, r \, dr \, dz. \tag{9.3}$$

Both approaches were implemented with trapezoid rule and give similar results.

10. Physics

Physical constants:

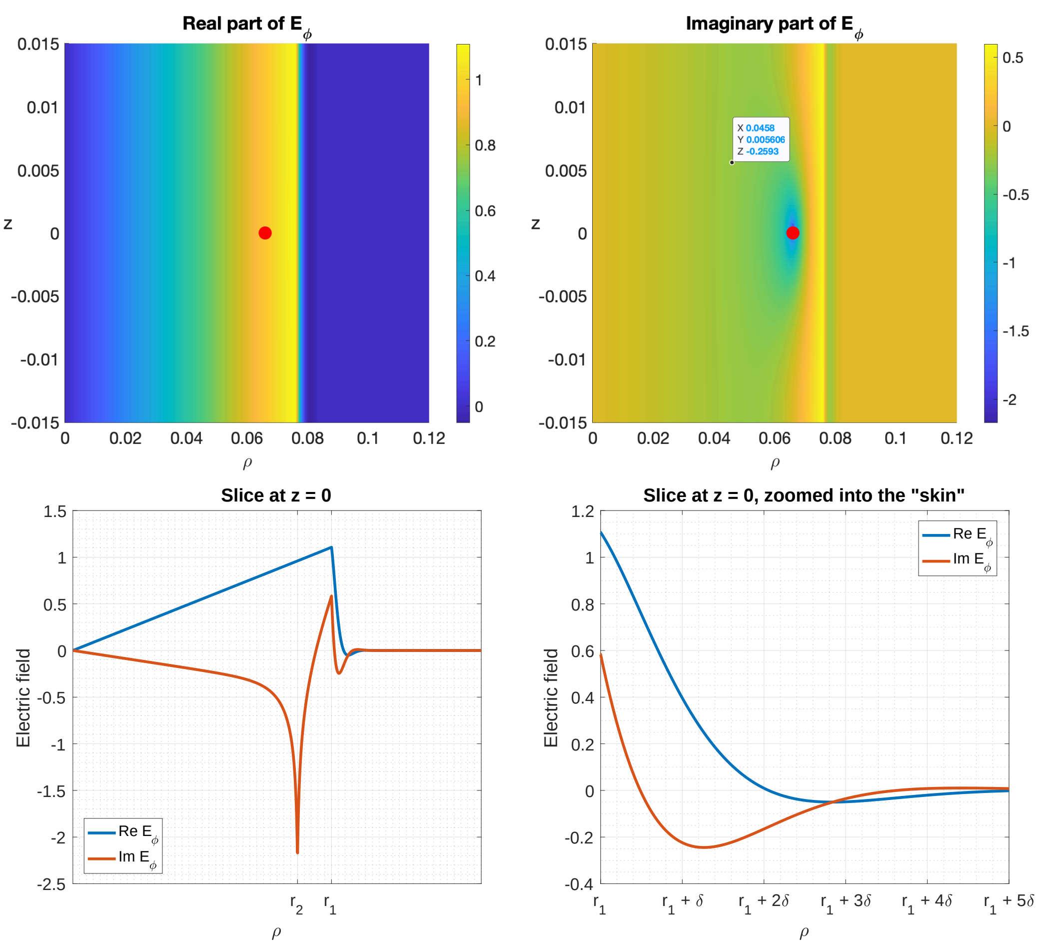
(1) Casing radius $r_1 = 76$mm.
(2) Source (coil) radius $r_2 = 66$mm.
(3) Half-pitch $L = 15$mm.
(4) $\omega = 2\pi \times 12$ kHz is the temporal frequency.

(5) Permeabilities[2]: $\mu_{\text{air}} = 1.26 \times 10^{-7} H/m$, $\mu_{\text{steel}} = 1.26 \times 10^{-6} H/m$
(6) Conductivity $\sigma_{\text{steel}} = 7 \times 10^{6}$ siemens/meter.
(7) Permittivities: $\epsilon_{\text{air}} = 9 \times 10^{-12} F/m$, $\epsilon_{\text{steel}} = \epsilon_{\text{air}} - \dfrac{i\sigma_{\text{steel}}}{\omega}$

With these values, the model gives the power flow into the steel casing of about 68 watts.

11. Plots

Here are a few plots generated by the accompanying Matlab script.



[2]The values of permeabilities are not those quoted for the "room temperature". They were adjusted to account for the effect of the high temperature.

12. Conclusion

We developed an analytical model which helps to compute the electric fields inside the heating element, as well as quantitatively assess its dependence on the physical parameters of the system. This analysis can be used to determine the optimal structure of the heating element, as well as to better understand the underlying physical processes.

A. Appendix : Jump Condition

To derive the jump condition we rewrite equation (6.6) as follows:

$$\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{x^2+1}{x}u = \frac{1}{\lambda_n}\delta\big(x/\lambda_n - r_2\big)$$

Integrate both sides in the neighbourhood of the jump:

$$\int_{\lambda_n r_2 - h}^{\lambda_n r_2 + h}\left(\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{x^2+1}{x}u\right)dx = \int_{\lambda_n r_2 - h}^{\lambda_n r_2 + h}\delta\big(x/\lambda_n - r_2\big)\frac{dx}{\lambda_n}$$

$$\left(\big(\lambda_n r_2 + h\big)\frac{\partial u}{\partial x}\Big|_{x=\lambda_n r_2 + h}\right) - \left(\big(\lambda_n r_2 - h\big)\frac{\partial u}{\partial x}\Big|_{x=\lambda_n r_2 - h}\right) + \mathcal{O}(h) = 1$$

Let $h \to 0$:

$$\left(\lambda_n r_2\frac{\partial u}{\partial x}\Big|_{x=\lambda_n r_2 +}\right) - \left(\lambda_n r_2\frac{\partial u}{\partial x}\Big|_{x=\lambda_n r_2 -}\right) = 1$$

$$\frac{\partial u}{\partial x}\Big|_{x=\lambda_n r_2 +} - \frac{\partial u}{\partial x}\Big|_{x=\lambda_n r_2 -} = \frac{1}{\lambda_n r_2}$$

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